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(Residential Autonomous College under University of Calcutta)

M.A./M.Sc. SECOND SEMESTER EXAMINATION, MAY 2014

FIRST YEAR

MATHEMATICS

Date : 21/5/2014

Time : 11 am – 1 pm

Paper : VI

Full Marks : 50

[Use a separate Answer Book for each Group]

Group – A

[*There are 7 questions carrying* ***45 marks****. Maximum you can score* ***30****.*]

1. Let p and q in $\mathbb{Q}$ satisfy $\sqrt{p} \notin \mathbb{Q}$ and $\sqrt{q} \notin \mathbb{Q}(\sqrt{p})$. Show that $\mathbb{Q}(\sqrt{p}, \sqrt{q}) = \mathbb{Q}(\sqrt{p}+\sqrt{q})$. Deduce that $x^4 - 2(p+q)x^2 + (p-q)^2$ is the minimal polynomial of $\sqrt{p}+\sqrt{q}$. [6]

2. Let $K \subseteq L \subseteq F$ be three fields such that [F:K] is finite. Prove that $[F:K] = [F:L][L:K]$. Hence show that if $f(x) = x^5 + 2x + 4$ is irreducible over $\mathbb{Q}$ and β is a zero of f(x) then none of $\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}$ belongs to $\mathbb{Q}(\beta)$. [7]

3. If F is a field such that the multiplicative group of units F* is cyclic, show that F is finite. [6]

4. a) Prove that $\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{2}\right) = \mathbb{Q}\left(\sqrt[6]{2}\right)$. [4]

   b) Describe all the finite subgroups of $\mathbb{C}^* = \mathbb{C} - \{0\}$. [2]

5. Prove that a field K is algebraically closed if and only if for every algebraic extension F/K we have F = K. Show that no finite field is algebraically closed. [7]

6. a) Let $\sqrt{3+4i}$ denote the square root of the complex number 3+4i that lies in the first quadrant and let $\sqrt{3-4i}$ denote the square root of the complex number 3 – 4i that lies in the fourth quadrant. Prove that : $\left[\mathbb{Q}\left(\sqrt{3+4i}+\sqrt{3-4i}\right) : \mathbb{Q}\right] = 1$. [3]

   b) Determine the degree of the extension $\mathbb{Q}\left(\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}\right)$ over $\mathbb{Q}$. [3]

7. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of odd degree and let u be a root of f(x). Prove that $\mathbb{Q}(u)$ has the property that –1 is not expressible as a sum of squares in $\mathbb{Q}(u)$. [7]

Group – B

[*Answer* ***any four*** *questions*]

8. Let α, β, γ be homomorphisms of the short exact sequence

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \to & A' & \to & B' & \to & C' & \to & 0 \end{array}$$

   Prove that if α and γ are isomorphisms then β is also an isomorphism where A, B, C, A', B', C' are R modules and R is commutative ring with 1. [5]

9. Let N be a submodule of M. Prove that if both M/N and N are finitely generated then M is also finitely generated. [5]

10. Show that $\mathbb{Q}[x]/(x^2-2)$ and $\mathbb{Q}[x]/(x^2-3)$ are isomorphic as modules over $\mathbb{Q}$ but not isomorphic as rings. [2+3]

11. Define free module and state its universal property. Show that $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}) \cong \mathbb{R}^n$. [1+1+3]

12. Prove that $\mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}\right) \cong \mathbb{Z}/(m,n)\mathbb{Z}$ where (m,n) = gcd (m,n). [5]

13. An element m of the R-module M is called a torsian element if rm = 0 for some nonzero element $r \in R$. Define $\mathrm{Tor}(M) = \{m \in M \mid rm = 0$ for same nonzero $r \in R\}$ Prove that if R is an integral domain then Tor (M) is a submodule of M. [5]